J. Fluid Mech. (1967), *vol.* 29, *part* 1, *pp.* 61–79
Printed in Great Britain

Imploding shocks and detonations

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(Received 17 March 1966 and in revised form 23 December 1966)

The gasdynamic problem of collapsing shocks and detonation waves having spherical or cylindrical symmetry is considered near the point or axis of symmetry. The solution basic to this work is the self-similar flow of a collapsing symmetrical shock wave with counterpressure neglected. The focusing effect as the flow progresses causes the front to accelerate and its velocity is singular at the instant of collapse. In the present work the perturbations, due to counterpressure and also to a uniform heat release, which give rise to essentially identical mathematical solutions, are evaluated. The basic self-similar solution is investigated in detail over a range of values of the specific heat ratio.

1. Introduction

The similarity solution to the problem of a contracting (imploding) spherical or cylindrical shock front propagating into a uniform gas at rest is well known. As the shock progresses its surface area diminishes, causing its velocity to increase towards the centre of symmetry, where it is infinite. The similarity solution is valid near the centre of symmetry, where the shock is strong.

In the present paper the shock is replaced by a contracting detonation front propagating into a uniform gas and releasing a constant amount of energy/unit mass of gas. At large distances from the centre, where the curvature is negligible, the detonation is a Chapman–Jouguet front, i.e. it travels with sonic speed relative to the burnt gas. The front accelerates towards the centre of symmetry and becomes overdriven, the motion now being governed more by the compression effects, due to focusing of the front, than by the heat release. The solution for the final stages is obtained as a perturbation, of order the inverse square of the speed of the front, on the corresponding similarity solution involving a shock wave. In the latter solution the strong shock relations are applied at the front so that only the undisturbed density enters into the problem, which has no time scale. In the present case of a detonation the heat release is taken into account, to first order so that the basic similarity hypothesis is unaltered, in the conservation equations at the front. The form of the perturbation so obtained is identical to that due to taking into account the pressure (or sound speed) of the undisturbed gas to first order. The disturbance of the speed of the front, due to heat release and initial pressure, is evaluated for several spherical and cylindrical cases by linearizing the equations of motion. The solution has to satisfy the conservation equations at the front and also be regular on a certain characteristic. The basic and perturbation equations are integrated numerically by making use of the power series expansions about this characteristic. A comparison is made with the results obtained by the approximate method as given by Whitham.

The results obtained by Butler for the Guderley solution are recomputed and extended. It is found necessary to investigate the existence and uniqueness of this solution.

The unsteady motion of a perfect, inviscid, non-heat-conducting gas is in general governed by partial differential equations. However, in the case of a flow which is one-dimensional, or spherically or cylindrically symmetric, so that the flow variables depend on a distance co-ordinate R and the time co-ordinate t, there is a class of solutions in which all variables are functions of a single combination of R and t, R/t^{α} where α is constant. Such flows are self-similar (Sedov 1959) and are governed by ordinary differential equations. The special case $\alpha = 1$ corresponds to a uniformly expanding or contracting flow, so that if such a flow is adiabatic then it is also homentropic, apart from entropy jumps across discontinuities, as any shock wave in the flow is of uniform strength. An example of a flow of this type ($\alpha = \frac{2}{5}$) is that of a strong point-explosion (Sedov 1959; Taylor 1950*b*) which involves an expanding, decaying spherical shock wave.

The problem to be investigated here is that of a contracting spherical or cylindrical detonation wave propagating into a uniform combustible gas. It is already known that there is no solution involving a uniformly contracting front (Selberg 1959; Stanyukovich 1960). This result will be deduced later from investigation of the integral curves of R/t, homentropic solutions.

In order to solve the problem of a contracting detonation front it will be necessary to study Guderley's solution (Guderley 1942; Butler 1954) for a converging shock wave, in which the shock front accelerates towards $R = 0$, where its velocity is infinite. If the shock path is $R = \lambda(t)$, then the shock speed U^* is given by

$$U^* \propto -\lambda^{1-1/\alpha}, \quad \text{where} \quad 0 < \alpha < 1.$$

The Guderley similarity solution is valid for small values of λ, for which the shock is strong so that the undisturbed gas pressure can be neglected. The flow variables behind the front thus depend upon the shock speed and undisturbed density only, which leads to the similarity hypothesis. If we now consider the effect of a uniform heat release in the medium as the front passes through it, then this results in the addition of a finite amount of energy/unit mass to the system and is thus a perturbation on the Guderley solution. The form of the perturbation can be deduced as follows. The particle velocity behind the shock is given by

$$u_s^* = \frac{2}{\gamma+1} U^* \propto -\lambda^{1-1/\alpha}.$$

Let the particle velocity behind the detonation be

$$u_D^* = u_s^* + V,$$

where V is supposed small relative to u_s^*. The extra kinetic energy/unit mass, which is directly due to the heat release and so must be finite, is of order Vu_s^* and hence the perturbation velocity V is of order U^{*-2} or $\lambda^{-2+2/\alpha}$ relative to the basic, shock wave solution. Similarly the sound speed perturbation is of order $\lambda^{-2+2/\alpha}$. Throughout the flow in general the perturbations are of order $R^{-2+2/\alpha}$. The effect of allowing for the initial pressure (or internal energy) of the undisturbed gas gives rise to perturbations of precisely the same form. Let the speed of the front be given by

$$U^* \propto -\lambda^{1-1/\alpha}(1+\beta\lambda^{-2+2/\alpha}),$$

where β is a constant due either to heat release or initial pressure, or both. In a given case we require the values of α, β to determine the path of the front.

The evaluation of the constant parameter α is performed by integrating the equations of motion, which can be reduced to a single first order, non-linear differential equation, subject to certain boundary conditions. In the case of a point-explosion α is determined simply by consideration of the dimensions of the basic parameters (the density and the energy of the explosion). However, in the contracting case there is only one basic parameter, the density, and a unique mathematical solution is obtained by assuming that the flow is regular on a certain characteristic following behind the shock. The conservation equations across the front and the regularity condition on the characteristic provide the two necessary boundary conditions for the solution of the differential equation. The values of α for the six cases $\gamma = 1{\cdot}2$, $1{\cdot}4$, $\frac{5}{3}$, spherical and cylindrical, have been computed by Butler. His results are extended to $\gamma = 3$, for the products of a detonation.

To find the correct value of α we must use a method of trial and error. However, the linearity of the perturbation equations means that the appropriate solution for β can be evaluated by taking a certain combination of any two linearly independent solutions. β is calculated for $\gamma = 1{\cdot}2$, $1{\cdot}4$, $\frac{5}{3}$, 3 for both cylindrical and spherical symmetry, and for heat release and undisturbed pressure. Comparison is made with results obtained by the approximate method in the form given by Whitham (1958). It is known that this approximate method, as applied by Chisnell (1957) in his 'shock-area' rule, gives extremely accurate results for the values of α but it is found here that the approximate values of β by this method are much less accurate.

The equations governing the motion are integrated between the front and the characteristic, which is necessary for the evaluation of α and β. To obtain the distribution of the physical variables behind the front the integration would have to be continued as far as $t = 0$, at which instant the shock is at $R = 0$ and is reflected. If all the heat energy available is released during the contracting phase of the motion then the front is reflected as a shock wave.

Contracting shock waves have previously been investigated both experimentally (Perry & Kantrowitz 1951) and numerically (Payne 1957). A problem having great similarity to that of converging shocks is that of cavitation in water, which has been studied by Hunter (1960, 1963) and differs from the former in the boundary conditions at the front and the fact that the motion is taken to be homentropic. A regularity condition on a certain characteristic is also employed to obtain a unique solution. The similarity hypothesis requires that the density in the cavity be zero. The effect of finite density (Holt & Schwartz 1963; Holt 1965; Holt, Kawaguti & Sakurai), to first order is that of a perturbation on

Hunter's solution, of order the inverse square of the speed of the front, and is analogous to the present work.

A perturbation of Guderley's solution due to departures from spherical symmetry has been studied by Butler (1956), who finds the collapse to be unstable in the presence of such disturbances.

2. Equations of motion and similarity

The equations governing the symmetric motion of a perfect, inviscid, non-heat-conducting gas with constant specific heats c_p, c_v, can be expressed in characteristic form as

$$\frac{\partial}{\partial t}(u^* \pm kc^*) + (u^* \pm c^*)\frac{\partial}{\partial R}(u^* \pm kc^*) = \mp\frac{ju^*c^*}{R} + \frac{1}{\gamma}c^{*2}\frac{\partial \phi^*}{\partial R}, \tag{1}$$

$$\frac{\partial \phi^*}{\partial t} + u^*\frac{\partial \phi^*}{\partial R} = 0, \tag{2}$$

where * denotes a physical quantity,

u^* denotes particle velocity,

c^* denotes sound speed, defined by $c^{*2} = (\partial p^*/\partial \rho^*)^*_s = \gamma p^*/\rho^*$,

s^* denotes specific entropy,

ϕ^*, a measure of entropy, is defined by $\phi^* = \log\left(\frac{c^{*\{2\gamma/(\gamma-1)\}}}{p^*}\right)$,

$k = \frac{2}{\gamma - 1}$,

and $j = 1$ for cylindrical symmetry, $j = 2$ for spherical symmetry.

Suppose that U^* is the velocity of a wave-front $R = \lambda(t)$, moving into uniform gas. For the case of a strong shock wave the boundary values immediately behind the front, which are identical to those for a plane front if λ is large in comparison with the shock width, are

$$\left.\begin{aligned} u^* &= 2/(\gamma+1)\,U^*, \\ \pm c^* &= [2\gamma(\gamma-1)]^{\frac{1}{2}}\,U^*/(\gamma+1), \\ \rho^*/\rho^*_0 &= (\gamma+1)/(\gamma-1), \\ \phi^* &= k\log(\pm U^*) + \phi^*_0, \end{aligned}\right\} \tag{3}$$

where ϕ^*_0 is the value of ϕ^* at some reference state and the negative sign is selected if U^* is negative. The assumption that the shock is strong leads to the neglect of the undisturbed pressure (or sound speed). Thus the flow behind the wave is determined by U^*, ρ^*_0, and since U^* has the dimensions of velocity it must be related to λ, t by $U^* = \alpha\lambda/t$, where α is a dimensionless constant and $t < 0$ for the contracting case ($t = 0$ is the instant at which the front is at $R = 0$). Hence

$$U^* = \frac{d\lambda}{dt} = \alpha\frac{\lambda}{t}$$

so that the equation of the front is

$$t = A\lambda^{1/\alpha}$$

and we can choose $A = -\alpha$ by fixing the length scale appropriately. Thus the front is

$$U^* = -\lambda^{1-1/\alpha}.$$

Let $\xi = t/\alpha R^{1/\alpha}$. The values of u^*, c^* on the shock, $\xi = -1$, are

$$u^* = -\frac{2}{\gamma+1}\lambda^{1-1/\alpha},$$
$$c^* = \frac{[2\gamma(\gamma-1)]^{\frac{1}{2}}}{\gamma+1}\lambda^{1-1/\alpha},$$
$$\phi^* = k\left(1-\frac{1}{\alpha}\right)\log\lambda + \phi_0^*$$

and the general values may be written

$$\left.\begin{aligned} u^* &= \frac{r(\xi)}{\xi}R^{1-(1/\alpha)}, \\ c^* &= \frac{s(\xi)}{\xi}R^{1-(1/\alpha)}, \\ \phi^* &= k\left(1-\frac{1}{\alpha}\right)\log R + \phi(\xi) + \phi_0^*, \end{aligned}\right\} \tag{4}$$

where r, s are respectively the dimensionless fluid velocity and sound speed and

$$r(-1) = \frac{2}{\gamma+1},$$
$$s(-1) = \frac{[2\gamma(\gamma-1)]^{\frac{1}{2}}}{\gamma+1},$$
$$\phi(-1) = 0.$$

This similarity form, in which the unknowns are functions of ξ only, may be substituted into the governing equations (1) and (2) to obtain ordinary differential equations for $r(\xi)$, $s(\xi)$, $\phi(\xi)$. The last of these is

$$\frac{d\phi}{d\xi} = \frac{k(1-\alpha)r}{1-r}, \tag{5}$$

which may be employed to eliminate $d\phi/d\xi$ from the first two, giving

$$\left.\begin{aligned} 2D\xi\frac{dr}{d\xi} &= (1-r+s)B_+ + (1-r-s)B_-, \\ 2kD\xi\frac{ds}{d\xi} &= (1-r+s)B_+ - (1-r-s)B_-, \end{aligned}\right\} \tag{6}$$

where

$$D = (r-1)(1-r+s)(1-r-s),$$
$$B_\pm = (r-1)\{1-\alpha(r\pm s)\}(r\pm ks) \mp j\alpha(r-1)rs + \frac{k(1-\alpha)}{\gamma}s^2.$$

The equations (6) combine to give a single differential equation for $r = r(s)$

$$\frac{1}{k}\frac{dr}{ds} = \frac{(1-r+s)B_+ + (1-r-s)B_-}{(1-r+s)B_+ - (1-r-s)B_-}. \tag{7}$$

Since the wave-front is at $R = 0$ at the instant $t = 0$, negative values of s, which correspond to negative values of t, arise from contracting fronts and positive values of s from expanding fronts.

The conservation equations across the front, assumed plane and including a heat release term are

$$\begin{gathered}\rho^*(u^* - U^*) = \rho_0^*(u_0^* - U^*),\\ p^* + \rho^*(u^* - U^*)^2 = p_0^* + \rho_0^*(u_0^* - U^*)^2,\\ \tfrac{1}{2}(u^* - U^*)^2 + \frac{\gamma}{\gamma - 1}\frac{p^*}{\rho^*} = \tfrac{1}{2}(u_0^* - U^*)^2 + \frac{\gamma}{\gamma - 1}\frac{p_0^*}{\rho_0^*} + Q,\end{gathered} \tag{8}$$

where Q is the heat release/unit mass of gas, $_0$ denotes the undisturbed gas and $u_0^* = 0$ if the gas is initially at rest.

The solution of (8) for u^*, c^*, ϕ^* in terms of Q, c_0^{*2} (retained to first order since $U^* \gg c_0^*, Q^{\frac{1}{2}}$) is

$$\left.\begin{aligned} u^* &= \frac{2}{\gamma + 1}U^* - KU^{*-1},\\ \pm c^* &= EU^* + E'U^{*-1},\\ \phi^* &= k\log(\pm U^*) + H_0 U^{*-2} + \phi_0^*,\end{aligned}\right\} \tag{9}$$

where
$$K = \frac{2}{\gamma+1}c_0^{*2} + (\gamma - 1)Q, \quad E = \frac{[2\gamma(\gamma-1)]^{\frac{1}{2}}}{\gamma + 1},$$
$$E' = \frac{6\gamma - \gamma^2 - 1}{2(\gamma+1)[2\gamma(\gamma-1)]^{\frac{1}{2}}}c_0^{*2} + \tfrac{1}{4}[2\gamma(\gamma-1)]^{\frac{1}{2}}(3-\gamma)Q,$$
$$H_0 = \frac{\gamma+1}{2(\gamma-1)}\left\{(\gamma+1)Q + \frac{3\gamma - 1}{\gamma(\gamma-1)}c_0^{*2}\right\}.$$

Thus the perturbation terms are of order U^{*-2} or $\lambda^{-2+2/\alpha}$ relative to the basic solution, as deduced previously, and the general solution is of the form

$$\left.\begin{aligned} u^* &= \frac{r(\xi)}{\xi}R^{1-(1/\alpha)} + \frac{\bar{r}(\xi)}{\xi}R^{-1+(1/\alpha)},\\ c^* &= \frac{s(\xi)}{\xi}R^{1-(1/\alpha)} + \frac{\bar{s}(\xi)}{\xi}R^{-1+(1/\alpha)},\\ \phi^* &= k\left(1 - \frac{1}{\alpha}\right)\log R + \phi(\xi) + F(\xi)R^{-2+(2/\alpha)}.\end{aligned}\right\} \tag{10}$$

The equations governing the system to zero and first order are obtained by examining the zero and first-order terms in (1), (2) substituting u^*, c^*, ϕ^* from the above. The zero-order equation (7), as obtained previously, and the first-order system, with s taken as the independent variable, are

$$(1 - r - s)\left(\frac{dr}{ds} + k\right)B_- = (1 - r + s)\left(\frac{dr}{ds} - k\right)B_+, \tag{11}$$

$$\left.\begin{aligned} (r-1)(1 - r \mp s)\left(\frac{d\bar{r}}{ds} \pm k\frac{d\bar{s}}{ds}\right)B_\pm &= A_\pm\left(\frac{dr}{ds} \pm k\right),\\ (r-1)\frac{dF}{ds}B_+ &= (1-\alpha)(1 - r - s)\left(\frac{dr}{ds} + k\right)\{2r(r-1)F + k\bar{r}\},\end{aligned}\right\} \tag{12}$$

where

$$\frac{A_\pm}{(r-1)^2(1-r\mp s)} = (1-r\mp s)(\bar{r}\pm k\bar{s})+(\bar{r}\pm\bar{s})\left\{\frac{B_\pm}{(r-1)(1-r\mp s)}-r\mp ks\right\}$$
$$\mp(1-\alpha)(k-1)(r\bar{s}-\bar{r}s)\mp ja(r\bar{s}+\bar{r}s)$$
$$+\frac{2k(1-\alpha)}{\gamma(r-1)}s\bar{s}-\frac{1-\alpha}{\gamma}\frac{s^2}{(1-r)^2}\{k\bar{r}+2(r-1)F\}.$$

3. Homentropic solutions

An examination of the dimensional parameters involved in the collapse of a spherical detonations front, namely heat release, and initial density and sound speed say, would suggest the possibility of a self-similar, uniform (R/t) collapse. In the expanding case, which is identical dimensionally, the uniformly expanding self-similar solution is well known (Taylor 1950*a*, *b*). In fact if these three quantities are taken as the basic parameters then a self-similar solution would have to be of the R/t type.

In the special case $\alpha = 1$, (7) reduces to

$$\frac{dr}{ds} = \frac{r}{s}\frac{(1-r)^2-s^2(1+j)}{(1-r)(1-r-jr/k)-s^2}.$$

The integral curves of this equation are given in figure 1 (Courant & Friedrichs 1948, p. 426), the direction being that of increasing time. The equation has six singular points:

$$(0,0),\quad (1,0),\quad (0,\pm 1),\quad \left(\frac{k}{j+k+1},\quad \frac{\pm[j+1]^{\frac{1}{2}}}{j+k+1}\right),$$

and it can be shown that the nature of these singularities does not depend on the value of γ or whether $j = 1$ or 2. A point in the (r, s)-plane corresponds to a path in the (R, t)-plane. The possible changes across a detonation or shock front form a locus in the (r, s)-plane. From the conservation equations across a detonation front, with a constant heat release Q/unit mass, the following relation between u^*, c^* behind the front can be obtained

$$c^{*2} = (U^*-u^*)\left\{\frac{\gamma-1}{2}u^*+U^*-(\gamma-1)\frac{Q}{u^*}\right\},$$

which, in terms of r, s, becomes

$$s^2 = (1-r)\left\{\frac{\gamma-1}{2}r+1-\frac{(\gamma-1)Q}{rU^{*2}}\right\}.$$

This is the equation of the locus of the possible transitions across a detonation front. The corresponding shock locus is obtained by setting $Q = 0$

$$s^2 = (1-r)\{\tfrac{1}{2}(\gamma-1)r+1\},$$

which is an ellipse. In each of these equations the value of r has to be not greater than $2/(\gamma+1)$, which corresponds to an infinitely strong front. The two curves intersect at $S\pm$ where $r = 2/(\gamma+1)$ as Q is negligible if U^* tends to infinity. The lines $r = 1\pm s$ are sonic lines and are critical in that the direction of integral curves changes on crossing them. Thus no physical solutions can cross $r = 1\pm s$. The detonation locus intersects these lines at $D\pm$, which are the Chapman–Jouguet detonation points. In the expanding case, $s > 0$, an integral curve runs from $D+$ to the point $(0, 1)$, which corresponds to a state of rest. This curve represents the solution given by Taylor's expanding, Chapman–Jouguet detonation wave.

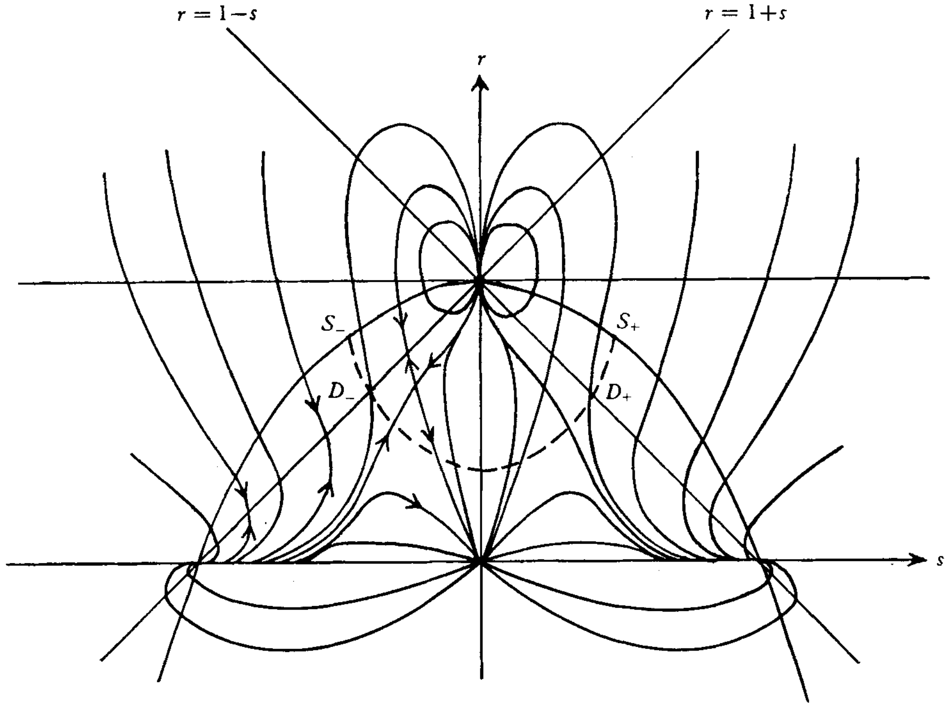

FIGURE 1

However, no integral curve can be extended from the point D_-, corresponding to a contracting Chapman–Jouguet detonation front. The arc $D_- S_-$ of the detonation locus corresponds to overdriven fronts, and integral curves intersecting this arc all run into the critical line $r = 1+s$. Hence there exists no uniformly contracting detonation fronts, either Chapman–Jouguet or overdriven.

Thus we have a mathematical argument for the non-existence of a uniformly collapsing shock as well as the physical one, that the focusing effect should cause the shock to accelerate towards the collapse point.

4. The limiting characteristic

The boundary values at the front for the basic solution are given by (3). However, the unknown parameter α appears in the differential equations, so that an extra condition remains to be found. This is obtained by examining the lines on which the solution of equations (5), (6) may be singular. There are four such lines:

$$1 - r \mp s = 0,$$
$$1 - r = 0,$$
$$\xi = -\infty.$$

The first pair gives the positive and negative characteristics through $R = 0, t = 0$, the third the corresponding particle path, and the fourth is the line $R = 0, t \geqslant 0$.

In the region $t < 0$ there is a limiting negative characteristic (l.n.c.) travelling behind the shock and reaching $R = 0$ at the same instant, $t = 0$, as the shock. For an arbitrary choice of α the solution will be singular on this line. Such a singularity could exist only if it were produced during the initiation of the shock and precisely on this limiting characteristic. For this reason we shall exclude the possibility of a singularity of this type and require the solution to be regular on the l.n.c. Let the equation of the l.n.c. be $\xi = \xi_1 (0 > \xi_1 > -1)$ in the basic flow, and

$$\xi = \xi_1(1+\delta R^{-2+2/\alpha}),$$

where δ is a constant, in the perturbed flow. Thus the boundary values of u^*, c^*, ϕ^*, in the form (10) on the l.n.c. are

$$u^* = u(\xi_1)\, R^{1-1/\alpha} + \{(du/d\xi)_{\xi_1}\, \xi_1\, \delta + \bar{u}(\xi_1)\}\, R^{-1+1/\alpha},$$

$$c^* = c(\xi_1)\, R^{1-1/\alpha} + \{(dc/d\xi)_{\xi_1}\, \xi_1\, \delta + \bar{c}(\xi_1)\}\, R^{-1+1/\alpha},$$

$$\phi^* = k\left(1-\frac{1}{\alpha}\right)\log R + \phi(\xi_1) + \left\{\left(\frac{d\phi}{d\xi}\right)_{\xi_1} \xi_1\, \delta + F(\xi_1)\right\} R^{-2+2/\alpha},$$

where $r = u\xi$, $s = c\xi$. Also, on this line

$$dR/dt = u^* - c^*$$

and, from its equation, we have

$$\frac{dR}{dt} = \frac{1}{\xi_1}\{R^{1-1/\alpha} - (3-2\alpha)\, R^{-1+1/\alpha}\}$$

on it. Equating the coefficients of $R^{1-1/\alpha}$, $R^{-1+1/\alpha}$ in these expressions for dR/dt gives

$$r_0 - s_0 = 1, \tag{13}$$

$$\bar{r}_0 = \bar{s}_0 + 2(\alpha-1)\,\delta + \frac{(r_1-1)\, B_{0+}}{2s_0^2(r_1+k)}\,\delta, \tag{14}$$

where

$$r_0 = r(s_0),$$

$$r_1 = (dr/ds)_{s=s_0},$$

where derivatives have been eliminated using the basic equations.

Since this line is a negative characteristic the variables there must satisfy the characteristic condition

$$d(u^*-kc^*) = \frac{jc^*u^*}{R}\, dt - \frac{c^*}{\gamma}\, d\phi^*$$

which, on setting the leading two coefficients zero, gives

$$\left(1-\frac{1}{\alpha}\right)(r_0 - ks_0) = jr_0\, s_0 - \frac{k}{\gamma}\left(1-\frac{1}{\alpha}\right) s_0, \tag{15}$$

$$\left\{\bar{s}_0 - \frac{\delta B_{0+}}{2s_0^2(r_1+k)}\right\}\left\{\frac{j\alpha}{\alpha-1}(r_0+s_0) + 1 - \frac{2(\gamma+1)}{\gamma(\gamma-1)}\right\}$$
$$+\delta s_0\left\{2j\alpha - 1 + \frac{2}{\gamma} + \frac{2k\alpha}{\gamma} + \frac{j\alpha(2\alpha-1)}{1-\alpha}\, r_0\right\} + \delta\left\{2\alpha - 3 - \frac{2k(1-\alpha)}{\gamma}\right\} + \frac{2}{\gamma}\, s_0 \bar{F}_0 = 0. \tag{16}$$

The conditions (15), (16) could have been derived directly from the differential equations. We require $dr/d\xi$, $ds/d\xi$ to be finite on $\xi = \xi_1$, i.e. $r = 1+s$. Hence we require $B_- = 0$ for $s = s_0$, which is equivalent to (15). Similarly, the condition that $d\bar{r}/d\xi$, $d\bar{s}/d\xi$ be finite for $s = s_0$ means that A_- must be zero to order $s-s_0$, which can be shown to be equivalent to (16).

5. The boundary conditions

The boundary values of u^*, c^*, ϕ^* at the front are given by (9), from which we can deduce the boundary values of r, s, $\bar{r}$, $\bar{s}$, F there, taking into account the displacement of the front from the shock path. The equation of the front is

$$\xi = -1+\frac{\beta}{3-2\alpha}\lambda^{-2+2/\alpha}.$$

Hence, on the front

$$\left.\begin{aligned} s &= -E, \\ r(-E) &= 2/(\gamma+1), \\ \bar{r}(-E) &= \frac{2\beta}{(\gamma+1)(3-2\alpha)}\left\{\frac{2j\alpha\gamma}{\gamma+1}+2\alpha-1\right\}-K, \\ \bar{s}(-E) &= \frac{-\beta E}{3-2\alpha}\left\{\frac{j\alpha(\gamma-1)}{\gamma+1}+\alpha+k-k\alpha\right\}-E', \\ F(-E) &= H_0+k\beta\left\{1+\frac{k(1-\alpha)}{3-2\alpha}\right\}. \end{aligned}\right\} \quad (17)$$

The above boundary values, together with the regularity conditions (14), (15), (16), serve to determine the solution. The basic solution for $r = r(s)$ has to satisfy the differential equation (7), which contains the unknown parameter α, and the boundary values $r(s_0) = r_0$, $r(-E) = 2/(\gamma+1)$, where r_0, s_0 are given in terms of α by (13), (15). On substituting (13), i.e. $r_0 = 1+s_0$, into (15) a quadratic in s_0 is obtained. Consider the expansion for $r(s)$ about the l.n.c.

$$r(s) = r_0+r_1(s-s_0)+r_2(s-s_0)^2+\ldots+r_n(s-s_0)^n+\ldots.$$

The solution can be developed theoretically by substituting this series into (7). The first equations, which determine r_0, r_1, are quadratic equations but all of the succeeding ones are linear. Thus, for a given value of α, there are four solutions and we require one of these solutions to pass through the shock point for some particular value of α.

The perturbation terms $\bar{r}$, $\bar{s}$, F have to satisfy the differential equations (12) together with the boundary conditions (17) containing the unknown parameter β, which measures the displacement of the front, and the boundary conditions (14), (16) containing the unknown l.n.c. displacement δ. The latter contains F_0, which is the boundary value of F on the l.n.c. and is the third unknown. Thus there are six boundary conditions containing three unknown parameters β, δ, F_0 and, since the boundary conditions are linear, the solution for $\bar{r}$, $\bar{s}$, F is uniquely determined in terms of $r(s)$.

6. The numerical solution

In order to evaluate the perturbations it will first be necessary to find the correct value of α and tabulate the basic solution $r = r(s)$ for the particular choice of γ and j. We can avoid the possible difficulty of dr/ds being indeterminate at $s = s_0$ on direct substitution into the differential equation (7) by making use of the power series expansion for $r(s)$ at $s = s_0$, the existence of which is ensured by the regularity assumption. However, direct computation of the coefficients r_n is out of the question because of the rapidly increasing complexity of the form of the equations for r_n, and each r_n has to be dealt with separately. For this reason the following iterative method is employed. Let R_n, R'_n be tabulated functions which represent r, r' respectively as far as the term involving r_n, in the form

$$R'_n = r_1 + 2r_2(s-s_0) + \ldots + nr_n(s-s_0)^{n-1} + (n+1)\,\epsilon_{n+1}(s-s_0)^n + O((s-s_0)^{n+1}),$$

$$R_n = r_0 + r_1(s-s_0) + \ldots + r_n(s-s_0)^n + \epsilon_{n+1}(s-s_0)^{n+1} + O((s-s_0)^{n+2}),$$

where ϵ_{n+1} is constant. From these we can deduce the next approximation R'_{n+1}, R_{n+1} by substituting the former into (7), written as $f(r, r', s) = 0$.

Then

$$f(R_n, R'_n, s) = (R_n - r)\frac{\partial f}{\partial r} + (R'_n - r')\frac{\partial f}{\partial r'} + \tfrac{1}{2}(R_n - r)^2\frac{\partial^2 f}{\partial r^2} + (R_n - r)(R'_n - r')\frac{\partial^2 f}{\partial r\,\partial r'} + \ldots,$$

where $\partial^2 f/\partial r'^2 = 0$, $(\partial f/\partial r')_0 = 0$, which we can write as

$$f(n) = (\epsilon_{n+1} - r_{n+1})\left\{\left(\frac{\partial f}{\partial r}\right)_0 + (n+1)\left(\frac{\partial f}{\partial r'}\right)_1\right\}(s-s_0)^{n+1} + O((s-s_0)^{n+2}).$$

In neglecting the term of order $(s-s_0)^{n+2}$ in the above we obtain a formula for $\epsilon_{n+1} - r_{n+1}$, and hence the following iterative formula for R'_{n+1}

$$R'_{n+1} = R'_n - \frac{(n+1)f(n)}{(s-s_0)\{(\partial f/\partial r)_0 + (n+1)(\partial f/\partial r')_1\}}. \tag{18}$$

The error coefficient ϵ_{n+2} in R'_{n+2} so obtained is independent of ϵ_{n+1} and is a function of n and the partial derivatives of f at $s = s_0$. From (20) we can tabulate R'_{n+1} throughout the range s_0 to $-E$. In practice the total range from s_0 to $-E$ is roughly 0·2 so that only 5 subdivisions of the range are sufficient to ensure that the integration does not introduce errors of order $(s-s_0)^{n+1}$ (otherwise the iteration would fail to converge to the solution). The initial approximations are taken as $R'_1 = r_1$, $R_1 = r_0 + r_1(s-s_0)$ and the iteration can be continued indefinitely.

The method can be extended to the solution of the three simultaneous equations (12) for $\bar{r}, \bar{s}, F$. Having selected δ, F_0 arbitrarily we find $\bar{r}_0$, $\bar{s}_0$ from (14) and (16). To form the initial approximations to $\bar{r}, \bar{s}, F$ we require their derivatives at $s = s_0$. These are obtained by equating to zero the appropriate coefficient in the expansions of the differential equations, which give rise to linear equations for $\bar{r}_1$, $\bar{s}_1$, F_1 so that the problem of choosing the appropriate solution does not arise here.

Thus we can tabulate $\bar{R}_1' = \bar{r}_1$, $\bar{R}_1 = \bar{r}_0 + \bar{r}_1(s-s_0)$, etc. The iterative formulae for $\bar{R}_{n+1}'1$, etc., are obtained by substituting the nth approximations into the governing equations and retaining only the first term, which gives three simultaneous, linear, algebraic equations for the corrections $\bar{R}_{n+1}' - R_n'$, etc. Solving the equations we obtain the required iterative formulae, and, for example

$$\bar{R}_{n+1}' = \bar{R}_n' - P\left[\left\{\left(\frac{\partial M}{\partial \bar{s}}\right)_1 + (n+1)\left(\frac{\partial M}{\partial \bar{s}'}\right)_2\right\} L(n) - (n+1)\left(\frac{\partial L}{\partial \bar{s}'}\right)_0 M(n)\,(s-s_0)^{-2} + \frac{N(n)(\partial M/\partial F)_1\,(\partial L/\partial \bar{s}')_0}{(\partial N/\partial F')_0}\right],$$

where (13) are denoted by L, M, N respectively,

$$M(n) = M(\bar{R}_n', \bar{R}_n, \bar{S}_n', \ldots),$$

$$P^{-1} = \left(\frac{\partial L}{\partial \bar{r}'}\right)_0 \left\{\left(\frac{\partial M}{\partial \bar{s}}\right)_1 + (n+1)\left(\frac{\partial M}{\partial \bar{s}'}\right)_2\right\} - \left(\frac{\partial L}{\partial \bar{s}'}\right)_0 \left\{\left(\frac{\partial M}{\partial \bar{r}}\right)_1 + (n+1)\left(\frac{\partial M}{\partial \bar{r}'}\right)_2\right\},$$

and $(\partial M/\partial \bar{r})_1$, for example, is the coefficient of $(s-s_0)$ in the expansion of $(\partial M/\partial \bar{r})$ about $s = s_0$, and is also the first non-vanishing coefficient.

7. The basic shock wave solution

For a given choice of γ, j, we wish to calculate the appropriate value of α and tabulate $r(s)$ from the l.n.c. to the front. It remains to be settled which of the four solutions can be made to satisfy the conditions of the problem. The six cases $\gamma = 1{\cdot}2$, $1{\cdot}4$, $\frac{5}{3}$ with $j = 1, 2$ were computed by Butler (1954). The same solution 'branch' is taken in each of these cases. In extending these results to the case $\gamma = 3$, corresponding to the motion of the products of a detonation, it is found that a different choice of branch is necessary. For this reason it was thought necessary to examine the behaviour of the integral curves of the differential equation with a view to examining the nature of the change-over and also the existence and uniqueness of the solution, particularly in the region of the changeover. The integral curves for the case $\gamma = 1{\cdot}4, j = 2$ are given by Guderley. The two cases selected here are $\gamma = \frac{5}{3}, j = 2$ and $\gamma = 3, j = 2$. The fact that these differ significantly suggests that a closer investigation is required.

The equation (7) for $r(s)$ has nine singular points. There are three on the r-axis $P_4(0,0)$, $P_1(0,1)$ and $(0,\alpha^{-1})$ and three in the region $s < 0$

$$P_2(s_{0+}, 1+s_{0+}), \quad P_3(s_{0-}, 1+s_{0-}), \quad P_5(S, k/\alpha(j+k+1)),$$

using Guderley's suffices. The remaining three are the mirror images of P_2, P_3, P_5 in the r-axis and correspond to expanding flows. The quantities $s_{0\pm}$ are the two roots of the quadratic for s_0 and S is the negative solution of

$$S^2 = r(r-1)(\alpha r - 1)/k\{\gamma(1-\alpha) + \alpha(1-r)\},$$

where $r = k/\alpha(j+k+1)$.

The behaviour of the integral curves is found by determining the nature of these singularities, the region of interest being $s < 0$, $0 < r < 1$. In these calculations the correct value of α was used. The curves are sketched in figure 2 for the case

$\gamma = \frac{5}{3}, j = 2$. All curves change direction on crossing the line $r = 1+s$, except for the two limiting ones through each of P_2 and P_3, which represent the four solutions which are regular on the l.n.c. We require a curve which starts at the shock point and passes through P_2 or P_3 and also through the origin P_4, which corresponds to $t = 0$. On this curve time must increase from the shock point to P_4. From the sketch it is seen that there are two such curves, one through P_2 and the other through P_3. One of these has to be made to pass through the shock point for some choice of α. For values of γ in this neighbourhood it was found in practice that an appropriate solution was found by selecting the curve through P_3 and the curve through P_2 could not be made to pass through the shock point.

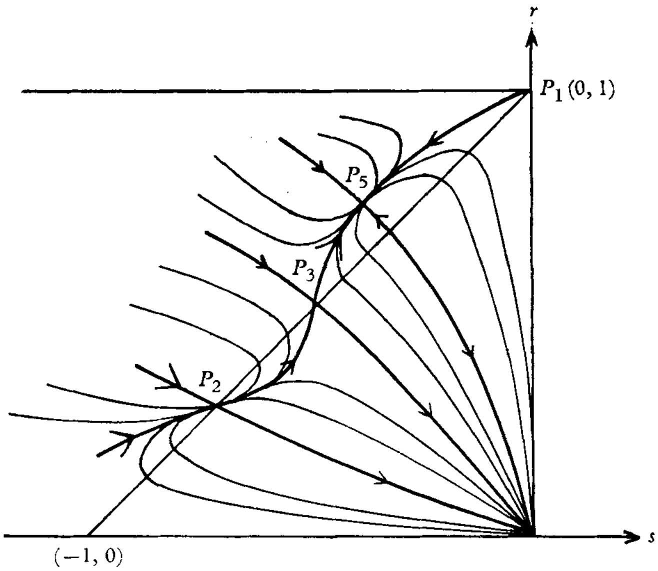

FIGURE 2

The sketch of the integral curves in the case $\gamma = 3, j = 2$ are given in figure 3. In this case P_2 and P_3 are both nodes and P_5 is a saddle point, below the line $r = 1+s$. Again only one solution was found, the curve through P_2 being slected in this case.

For given γ, j there is a range of values $\alpha_1 < \alpha < \alpha_2$ for which s_0 is imaginary. The range $0 < \alpha < \alpha_1$ never yields any solutions. For $j = 2$, as γ is increased from 1·2, the correct value of α approaches α_2 and the correct values of s_0 approach each other. For some critical value of γ, γ_c say, the roots are equal and the transition from one branch to the other occurs at γ_c. For any given value of γ, α_2 is that value of α for which P_2, P_3 coincide. For values of $\alpha < \alpha_2$ these two singular points are complex and so no regular solutions can be continued across the line $r = 1+s$ to the origin. As α is increased from α_2, P_2 and P_3 separate and move along the segment of $r = 1+s$ as far as $(0, 1)$, $(-1, 0)$ when $\alpha = 1$.

We require a solution through either P_2 or P_3, the solution and the positions of the two points depending on the value of α, and also through the shock point, the position of which depends on γ only. In table 1 $d(P_2)$ denotes the discrepancy, for the spherical case, between the solution obtained by integrating from P_2 as far as $s = -E$ and the required value of $2/(\gamma+1)$ there. For $\gamma = 1{\cdot}865$ $d(P_3)$ has a zero in the given range and this zero corresponds to the actual solution. Apparently $d(P_2)$ has no zero. The situation is reversed in the case $\gamma = 1{\cdot}875$, the point P_2 being appropriate in this case. Thus, $1{\cdot}875 > \gamma_c > 1{\cdot}865$. The transition at $\gamma = \gamma_c$ takes place smoothly and there is no apparent physical significance to the case $\gamma = \gamma_c$.

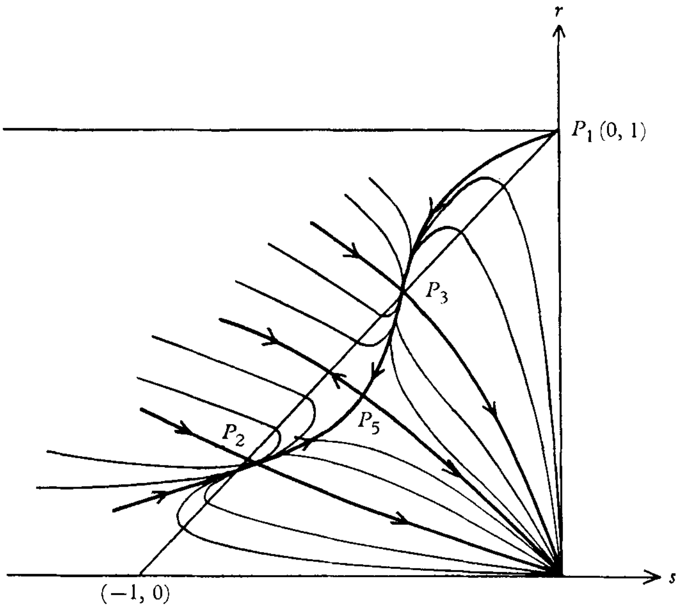

FIGURE 3

$\gamma = 1{\cdot}865$			$\gamma = 1{\cdot}875$		
α	$d(P_2)$	$d(P_3)$	α	$d(P_2)$	$d(P_3)$
0·674453	s_0 imag.	s_0 imag.	0·6738558	s_0 imag.	s_0 imag.
0·6744535	−0·00459	−0·00033	0·6738559	+0·00206	+0·0033
0·674454	−0·00557	+0·00066	0·6738560	+0·00132	+0·0039
0·674456	−0·00789	+0·00308	0·6738562	+0·00079	+0·0046
0·674460	−0·0109	+0·00606	0·67386	−0·00387	+0·0092
0·675	−0·0749	+0·0751	0·675	−0·1003	+0·118

TABLE 1

For a given γ the roots for s_0 are monotonic in α in the range $\alpha_2 < \alpha < 1$, so that P_2, P_3 vary continuously, without repetition, along the arc $r = 1+s$ as α varies between α_2 and 1. Together with the results of table 1 this suggests the following behaviour. For $\alpha = \alpha_2$, P_2 and P_3 coincide and the single integral curve through them separates the area $0 < r < 1$, $r > 1+s$ into two distinct regions. As α is increased the two integral curves through P_2, P_3 must lie wholly within each of these regions so that points in the lower region may be reached from P_3 for some value of α and those above from P_2. It seems likely that no two curves through one of P_2, P_3 will intersect for distinct choices of α. If this is so then the solution will be unique for all γ. The choice between P_2, P_3 is determined by whether the shock point lies above the limiting integral curve through the point formed by the merging of P_2, P_3. Apparently for $\gamma < \gamma_c$ the shock point lies below this curve, and above it for $\gamma > \gamma_c$.

The results for the eight cases $\gamma = 1{\cdot}2$, $1{\cdot}4$, $\frac{5}{3}$, 3 with $j = 1, 2$ are given in table 2 along with those given by Whitham's approximate method, to be described later, for comparison.

	$j = 1$		$j = 2$	
γ	α	α approx.	α	α approx.
1·2	0·861163	0·859762	0·757142	0·754021
1·4	0·835323	0·835373	0·717174	0·717288
$\frac{5}{3}$	0·815625	0·816043	0·688377	0·688654
3	0·775667	0·772661	0·636411	0·629542

TABLE 2

8. The perturbation solution

The solution for the perturbations are now obtained by integrating the three simultaneous equations (11) for $\bar{r}$, $\bar{s}$, F, subject to the boundary conditions at the front and the l.n.c. The function $r(s)$ and the parameter α appearing in (11) are now known. As for the basic solution we develop the solution away from the l.n.c., having satisfied the regularity condition there, as far as the front. To do so we select arbitrary values of δ, F_0, which determine $\bar{r}$, $\bar{s}$, F at $s = s_0$, and continue the solution to $s = -E$, where the conditions will, in general, not be satisfied by the present solution. Suppose we have found two such linearly independent solutions, corresponding to choices $\delta^{(0)}$, $F_0^{(0)}$ and $\delta^{(1)}$, $F_0^{(1)}$ for the values of δ, F_0. Let $\bar{r}_0^{(0)}$, $\bar{r}_H^{(0)}$ denote the values of $\bar{r}$ at $s = s_0$, $-E$ respectively of the $_0$ solution. The boundary conditions at the front, given by (19), can be written as

$$\begin{aligned} \bar{r}_H &= A_1\beta - K, \\ \bar{s}_H &= A_2\beta - E', \\ F_H &= A_3\beta + H_0, \end{aligned}$$

where β is unknown. Let us take a linear combination of the two numerical solutions and satisfy the above conditions. Thus

$$\left.\begin{aligned} X\bar{r}_H^{(0)} + Y\bar{r}_H^{(1)} &= A_1\beta - K, \\ X\bar{s}_H^{(0)} + Y\bar{s}_H^{(1)} &= A_2\beta - E', \\ XF_H^{(0)} + YF_H^{(1)} &= A_3\beta + H_0, \end{aligned}\right\} \tag{19}$$

which can be readily solved by X, Y, β so that we can thus evaluate β. The solution for $\bar{r}$, $\bar{s}$, F appropriate to the boundary conditions can be obtained by performing the integration from the l.n.c. with the correct values of δ, F_0 which are given by

$$F_0 = XF_0^{(0)} + YF_0^{(1)},$$

$$\delta = \frac{X(\bar{r}_0^{(0)} + \bar{s}_0^{(0)}) + Y(\bar{r}_0^{(1)} + \bar{s}_0^{(1)})}{2(\alpha - 1) + [(r_1 - 1)B_{0+}/2s_0^2(r_1 + k)]}.$$

The quantities K, E', H_0 appearing in (19) depend upon the values of Q, c_0^{*2}. However, the dependence is linear so that all that is necessary is to evaluate two solutions due to linearly independent choices of Q, c_0^{*2}. For simplicity we can take $Q = 1, c_0^{*2} = 0$ and $Q = 0, c_0^{*2} = 1$, the former corresponding to a detonation front and the latter to the correction due to counter-pressure. The solution in a specific

γ	β approx.	β	$\beta-\frac{1}{2}(\gamma+1)K$	$\beta+E'/E$
1·2	−0·0482	−0·1009	−0·3209	0·8891
1·4	0·2158	0·2508	−0·2292	1·2108
$\frac{5}{3}$	0·5894	0·7737	−0·1152	1·6626
3	3·2679	4·4760	+0·4760	4·4760

TABLE 3. $j = 2$, $Q = 1$, $c_0^{*2} = 0$

γ	β approx.	β	$\beta-\frac{1}{2}(\gamma+1)K$	$\beta+E'/E$
1·2	−0·04816	−0·08199	−0·3020	0·9080
1·4	0·2158	0·2310	−0·2490	1·1910
$\frac{5}{3}$	0·5894	0·6692	−0·2995	1·4783
3	3·268	3·594	−0·4056	3·5944

TABLE 4. $j = 1$, $Q = 1$, $c_0^{*2} = 0$

γ	β approx.	β	$\beta-\frac{1}{2}(\gamma+1)K$	$\beta+E'/E$
1·2	−0·4730	−0·7047	−1·7047	4·2536
1·4	0·2172	0·3097	−0·6903	2·7382
$\frac{5}{3}$	0·4541	0·6942	−0·3058	2·0942
3	0·6667	1·0435	+0·0435	1·3769

TABLE 5. $j = 2$, $Q = 0$, $c_0^{*2} = 1$

γ	β approx.	β	$\beta-\frac{1}{2}(\gamma+1)K$	$\beta+E'/E$
1·2	−0·4730	−0·6211	−1·6212	4·3371
1·4	0·2172	0·2593	−0·7407	2·6879
$\frac{5}{3}$	0·4541	0·5587	−0·4413	1·9875
3	0·6667	0·7738	−0·2262	1·1071

TABLE 6. $j = 1$, $Q = 0$, $c_0^{*2} = 1$

case, due to either or both of these effects, is found by taking the appropriate combination of these two solutions. For $\gamma = 1{\cdot}2$, $1{\cdot}4$, $\frac{5}{3}$, 3 and $j = 1, 2$ we have sixteen distinct cases. The results for these are given in tables 3, 4, 5 and 6 along with those obtained by the approximate method of Whitham. The boundary values of u^*, c^* at the front are given by

$$u^* = -\frac{2}{\gamma+1}\lambda^{1-1/\alpha}\left\{1+\left(\beta-\frac{\gamma+1}{2}K\right)\lambda^{-2+2/\alpha}\right\},$$

$$c^* = E\lambda^{1-1/\alpha}\left\{1+\left(\beta+\frac{E'}{E}\right)\lambda^{-2+2/\alpha}\right\}.$$

The coefficients $\beta-\frac{1}{2}(\gamma+1)K$, $\beta+E'/E$ are also tabulated.

9. The results and the Whitham simplified analysis

Before discussing the results it will be of interest to evaluate the solution by the approximate method in the form given by Whitham. It is known that this gives remarkable accuracy in estimating α (Chisnell 1957; Whitham 1958). Chisnell employed his 'shock-area' rule, which he formulated for shock waves in channels of slowly varying cross-section, in the evaluation of α for $\gamma = 1{\cdot}2,\ 1{\cdot}4,\ \frac{5}{3}$, $j = 1, 2$ and compared the results with Butler's. Whitham obtained Chisnell's results by assuming that the characteristic conditions to be satisfied behind the shock will be satisfied by the boundary values there. This method will be applied to the present problem.

The characteristic condition to be satisfied behind the front is

$$d(u^* - kc^*) = ju^*c^*R^{-1}\,dt - \gamma^{-1}c^*\,d\phi^*$$

and the boundary values there are

$$\begin{aligned} u^* &= -\frac{2}{\gamma+1}R^{1-1/\alpha} + \left(K - \frac{2\beta}{\gamma+1}\right)R^{-1+1/\alpha}, \\ c^* &= ER^{1-1/\alpha} + (E' + \beta E)\,R^{-1+1/\alpha}, \\ \phi^* &= k(1-1/\alpha)\log R + (H_0 + k\beta)\,R^{-2+2/\alpha}. \end{aligned}$$

On substituting these values into the characteristic condition, and using the fact that $dR/dt = u^* - c^*$, we obtain a polynomial in $R^{-2+2/\alpha}$. Setting the first term zero gives Whitham's formula for α

$$\frac{1}{\alpha} - 1 = \left(\frac{jE}{\gamma+1}\right)\Bigg/\left(\frac{2}{\gamma+1} + E\right)\left(\frac{1}{\gamma+1} + \frac{E}{\gamma}\right). \tag{20}$$

Equating the coefficient of the second term to zero yields

$$\beta = \left(E^2K - \frac{4}{(\gamma+1)^2}E'\right)\Bigg/\frac{4}{\gamma+1}E\left(\frac{2}{\gamma+1} + E\right) + \left(\frac{2}{\gamma}EH_0 + K - \frac{2}{\gamma\mu}E'\right)\Bigg/4\left(\frac{1}{\gamma+1} + \frac{E}{\gamma}\right). \tag{21}$$

These approximate results neglect the effect of disturbances reaching the shock from behind, due to the characteristic condition not being applied correctly. The changing surface area of the shock is accounted for. The area of the front is proportional to R^j which results in the exponent in the power law for the shock speed, i.e. $1 - 1/\alpha$, being proportional to j. The perturbation solution for β, which arises from energy terms proportional to volume and independent of the geometry of the system, is independent of j. The result (20) for α is very accurate because the propagation of the shock is largely governed by the focusing effect, due to its surface area diminishing, and is affected little by other disturbances. This is not the case for the perturbations, neither of which (heat release and initial pressure) are geometric effects and the results for β are much less accurate than those for α. A graph of the approximate results for β is given in figures 4 and 5.

From the results obtained by the full analysis it is seen that for given j, Q, c_0^{*2} there is always a change in sign in β, considered as a function of γ. Thus the introduction of either of the two effects can produce an increase or a decrease in the unscaled speed of the front, depending upon the value of γ. In each case $\beta = 0$ for some value of γ and in this case the perturbation of the front speed is of order $\lambda^{-4+4/\alpha}$. The critical values of γ were found by the approximate method to be 1·30 for $Q = 0$, $c_0^{*2} = 1$ and 1·24 for $Q = 1$, $c_0^{*2} = 0$. The graph of β from the approximate method follows the behaviour of the correct values fairly closely, and the former would appear to be sufficiently accurate to estimate the critical values of γ to two decimal places.

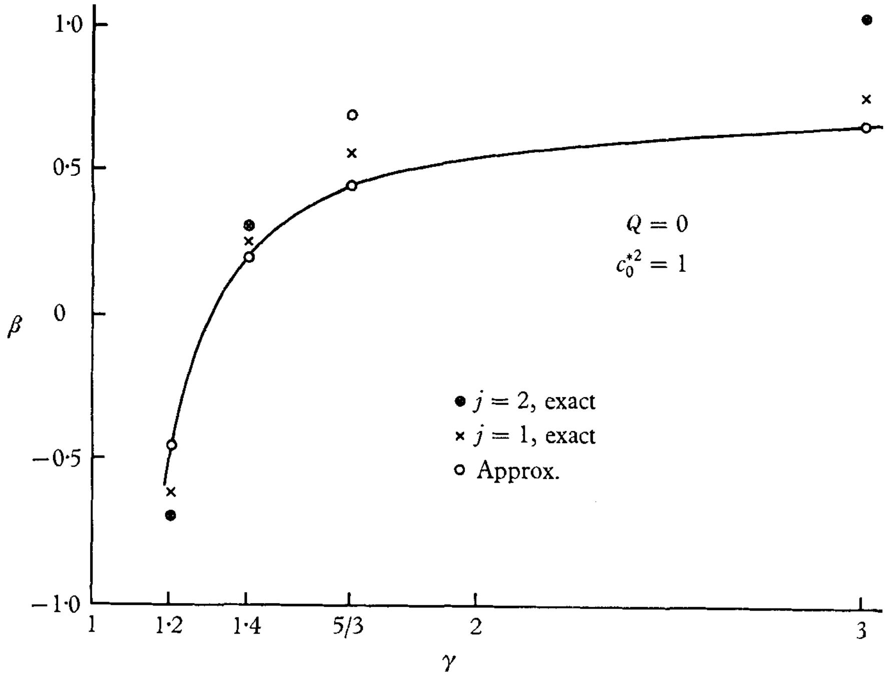

FIGURE 4

The occurrence of negative values for β does not necessarily mean that the addition of a heat release in a given case would cause the front to slow down. The results of this paper would have to be scaled for a given situation. This scaling would be dependent on the initial conditions, which are necessarily excluded from the present analysis. Also no simple overall energy consideration may be applied as the wave has in effect come from infinity. This is not the case for expanding self-similar flows where the region of validity is the total region in motion.

If the initially uniform medium considered so far is replaced by a medium initially at rest but having variable density, $\rho_0^* \propto R^m$, say where m is a constant, then the initial sound speed $c_0^* \propto R^{-m/2}$. The similarity hypothesis will still hold provided c_0^* remains small relative to U^*, i.e. $\frac{1}{2}m < 1/\alpha - 1$, and the solution is

$$U^* = -\lambda^{1-1/\alpha}(1 + \beta\lambda^{-1+1/\alpha-m/2}),$$

where the value of β is identical to that in the case $c_0^{*2} = 1$, $Q = 0$, computed previously. The coefficients of the perturbations do not differ from the previous solution, the only difference being in the power of λ.

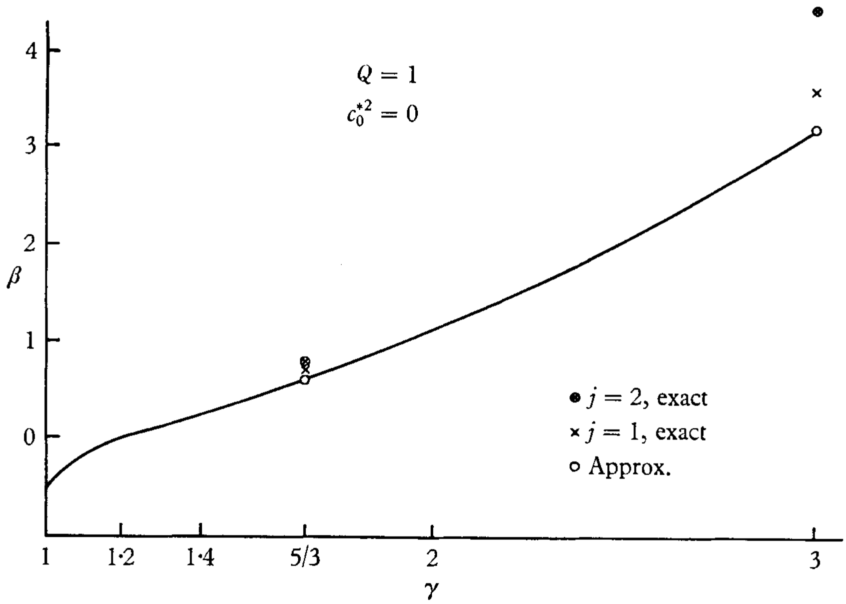

FIGURE 5

The author gratefully acknowledges the guidance of Mr D. S. Butler throughout the course of this work and in particular for suggesting the numerical solution. The author is also grateful to Prof. D. C. Pack for suggesting the problem. The computations were performed on the Atlas Computer of the Science Research Council, Didcot, Berkshire. The work was completed in its present form while the author was in the Division of Aeronautical Sciences, University of California, Berkeley, under the support of Office of Naval Research Contract no. GR 25866.

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